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CONJUGATE PROBLEMS OF UNSTEADY HEAT AND MASS CONDUCTION UNDER VARYING EXTERNAL CONDITIONS

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A closed system of algebraic and ordinary differential equations that enables one to simply and with a high degree of accuracy solve unsteady problems of heat conduction and diffusion extraction of a substance from solids of standard shapes – a plate, a cylinder, and a sphere – under varying external conditions is proposed. The method of solution is based on a unified universal dependence that describes with a high degree of accuracy the distribution of temperatures or concentrations of the substance in the above solids and on equations that determine variations in characteristics involved in this dependence with time and space as the solids move.

1. The problems of unsteady heat and mass conduction in a plate, a cylinder, and a sphere are among classical problems of mathematical physics, and widely known methods of solving them are presented in numerous monographs, handbooks, and textbooks. The solutions of these problems amount to infinite series with a very cumbersome structure of each term and with complex algorithms for finding the coefficients of the series. And this presents certain difficulties both in solution and physical interpretation of the relations. Furthermore, external and boundary conditions are taken to be constant or vary in accordance with the simplest strictly specified laws whose number is no greater than four. Although formally classical solutions refer to exact solutions, they are all indiscriminately realized as approximate solutions. In this connection, it is logical to seek less cumbersome approximations, however, of a sufficient degree of accuracy, especially as far as the rigor of satisfying the boundary conditions is concerned. The possibility of this approximation existing is predetermined by parameter distributions inside the enumerated solids. These distributions are continuous, smooth, symmetric, and monotonic in the segments $[0, Z]$. As in classical methods, we will seek a solution in the form of a power series, the latter, however, in the form of an interpolation rather than an extrapolation polynomial, which will enable us to substantially limit the number of terms of the series. This approach was successfully used to solve complex aerohydrodynamical problems [1, 2].

2. Classical diffusion problems in a spatially one-dimensional statement are reduced to solving the equation

$$\frac{dT}{dt} = a\left(\frac{\partial^2 T}{\partial z^2} + \frac{i-1}{z}\frac{\partial T}{\partial z}\right) \tag{1}$$

for a plate $(i = 1)$, a cylinder $(i = 2)$, and a sphere $(i = 3)$. The left-hand side of (1) here is written as the substantial derivative

$$\frac{dT}{dt} \equiv \frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y}, \tag{2}$$

where v is the velocity of motion of the solid in the direction y. Out of the two forms of the right-hand side of (2), use is usually made of one form. They can be mutually transformed. In the vicinity of the line (point) of symmetry, according to the l'Hôpital rule, Eq. (1) takes the form

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$$\lim_{z\to 0} \frac{dT}{dt} = ia \left(\frac{\partial^2 T}{\partial z^2}\right)_0 \tag{3}$$

Then, for all i, according to (1) and (3), we obtain

$$\left(\frac{\partial^3 T}{\partial z^3}\right)_0 = 0 . \tag{4}$$

For the boundary conditions of the third kind, the parameters on the line (at the point) of symmetry and at the boundary of the solid will be:

$$\begin{aligned} z = 0&: \ T_1 = T_{10} , \ \frac{\partial T_1}{\partial z} = 0 , \ \frac{\partial^3 T_1}{\partial z^3} = 0 ; \\ z = Z&: \ T_1 - T_{1S} , \ \frac{\partial T_1}{\partial z} = \frac{\alpha_2}{\lambda_1} (T_{20} - T_{2S}) , \end{aligned} \tag{5}$$

where T_{10}, T_{1S}, and T_{2S} are the sought functions.

Let us represent T_1 in the form of a four-power interpolation polynomial with the variable z:

$$T_1 = \sum_{k=0}^{4} a_k z^k , \tag{6}$$

where, according to (5), expressions for the coefficients a_k are as follows:

$$\begin{aligned} a_0 &= T_{10} , \\ a_1 &= 0 , \\ a_2 &= -\frac{1}{Z^2} [2 (T_{10} - T_{1S}) - 0.5 \text{ Bi} (T_{2S} - T_{20})] , \\ a_3 &= 0 , \\ a_4 &= \frac{1}{Z^4} [(T_{10} - T_{1S}) - 0.5 \text{ Bi} (T_{2S} - T_{20})] , \\ \text{Bi} &= \frac{\alpha_2 Z}{\lambda_1} . \end{aligned} \tag{7}$$

It is easy to transform dependence (6) to the formula of the excess temperature distribution in the solid

$$\frac{T_1 - T_{1S}}{T_{10} - T_{1S}} = (1 - \eta^2)^2 + 0.5A (1 - \eta^2) \eta^2 , \tag{8}$$

where

$$\eta = \frac{z}{Z} ; \ A = \text{Bi} \frac{T_{2S} - T_{20}}{T_{10} - T_{1S}} . \tag{9}$$

The parameter A is the ratio of the gradient T_1 on the solid surface to the average gradient T_1 in the solid. At the beginning and in the end of the process when $T_{10} \approx T_{1S}$ the parameter A tends to increase without limit ($A \to \infty$). By testing (8) for an extremum it is established that this formula can be used when

$$A \le A_{\mathrm{cr}} = 4\,, \tag{10}$$

while, for $A = A_{\mathrm{cr}}$, formula (8) itself takes the form

$$\left(\frac{T_1 - T_{1S}}{T_{10} - T_{1S}}\right)_{\mathrm{cr}} = 1 - \eta^4 = 1 - \eta^{A_{\mathrm{cr}}}\,. \tag{11}$$

Therefore when $A > A_{\mathrm{cr}}$ there is a smooth transition in the formula

$$\frac{T_1 - T_{1S}}{T_{10} - T_{1S}} = 1 - \eta^A\,, \tag{12}$$

that satisfies the axial and boundary conditions (5) when $A \ge A_{\mathrm{cr}}$.

Thus, we obtained the simple unified universal one-parametric dependences (8), (9), and (12) that describe the distributions of T_1 in a plate, a cylinder, and a sphere.

3. Conditions (5) predetermine a priori the high accuracy of distributions (8) and (12). To be more convincing, we present below comparison of the calculations by formulas (8) and (12) with classical calculations based on the Fourier series and the Bessel functions. The data given in the handbook [3] at pp. 137-148, retaining the main notation adopted in this publication, are taken as basic data. The basic dependence (2.40) [3] has the form

$$\frac{T_1 - T_2}{T_{01} - T_2} = \sum_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot \bar{\xi}) \exp(-\mu_n^2 \mathrm{Fo})\,, \tag{13}$$

where T_{01} is the value of T_1 at the initial instant while $\bar{\xi} \equiv \eta$. According to (13), the transition to excess distribution of T_1 (as the left-hand sides of (8) and (12)) will be

$$\frac{T_1 - T_{1S}}{T_{10} - T_{1S}} = \frac{\sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot \bar{\xi}) \exp(-\mu_n^2 \mathrm{Fo}) - \sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot 1) \exp(-\mu_n^2 \mathrm{Fo})}{\sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot 0) \exp(-\mu_n^2 \mathrm{Fo}) - \sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot 1) \exp(-\mu_n^2 \mathrm{Fo})}\,, \tag{14}$$

while the parameter A in accordance with (9) will be determined as

$$A = \mathrm{Bi} \frac{\sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot 1) \exp(-\mu_n^2 \mathrm{Fo})}{\sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot 0) \exp(-\mu_n^2 \mathrm{Fo}) - \sum\limits_{n=1}^{\infty} A(\mu_n)\, U(\mu_n \cdot 1) \exp(-\mu_n^2 \mathrm{Fo})}\,. \tag{15}$$

Here we set

$$T_{2S} = T_{1S} \tag{16}$$

(the condition $T_{2S} \neq T_{1S}$ is considered below when the dependences are extended to the case of diffusion of a substance from capillary-porous bodies). According to the recommendations of [3], for $\mathrm{Fo} \ge 0.3$, it is allowable to use only the first term of series (13). In this case, (14) and (15) are somewhat simplified:

TABLE 1. Comparison of the Distributions of Dimensionless Temperatures according to [3] (T) and by the Method Proposed (T_m) in a Plate

Parameters	Parameters for different ξ									Bi	A
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9		
T	0.9900	0.9600	0.9099	0.8399	0.7499	0.6398	0.5098	0.3598	0.1899	0.01	1.9940
T_m	0.9900	0.9599	0.9098	0.8396	0.7494	0.6393	0.5092	0.3593	0.1895		
T_m/T	1.0000	0.9999	0.9998	0.9996	0.9994	0.9992	0.9989	0.9986	0.9982		
T	0.9899	0.9597	0.9093	0.8389	0.7485	0.6381	0.5080	0.3581	0.1888	0.10	1.9735
T_m	0.9899	0.9595	0.9089	0.8382	0.7475	0.6369	0.5067	0.3569	0.1880		
T_m/T	0.9999	0.9998	0.9995	0.9992	0.9987	0.9981	0.9975	0.9967	0.9958		
T	0.9894	0.9575	0.9048	0.8314	0.7381	0.6254	0.4942	0.3455	0.1803	1.0	1.8692
T_m	0.9894	0.9575	0.9046	0.8312	0.7377	0.6249	0.4937	0.3449	0.1799		
T_m/T	1.0000	0.9999	0.9999	0.9997	0.9996	0.9993	0.9989	0.9984	0.9977		
T	0.9881	0.9528	0.8946	0.8149	0.7152	0.5975	0.4644	0.3184	0.1625	10.0	1.6636
T_m	0.9883	0.9535	0.8962	0.8174	0.7185	0.6012	0.4680	0.3212	0.1641		
T_m/T	1.0002	1.0008	1.0018	1.0031	1.0046	1.0062	1.0078	1.0091	1.0098		
T	0.9877	0.9513	0.8914	0.8097	0.7080	0.5889	0.4552	0.3101	0.1571	100.0	1.5962
T_m	0.9880	0.9522	0.8935	0.8129	0.7121	0.5935	0.4595	0.3135	0.1589		
T_m/T	1.0003	1.0010	1.0023	1.0039	1.0058	1.0077	1.0096	1.0109	1.0114		

TABLE 2. Comparison of the Distribution of Dimensionless Temperatures according to [3] (T) and by the Method Proposed (T_m) in a Sphere

Parameters	Parameters for different ξ									Bi	A
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9		
T	0.9900	0.9599	0.9099	0.8398	0.7497	0.6397	0.5096	0.3597	0.1898	0.01	1.9960
T_m	0.9900	0.9599	0.9098	0.8397	0.7496	0.6395	0.5095	0.3595	0.1897		
T_m/T	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996		
T	0.9899	0.9594	0.9088	0.8380	0.7472	0.6366	0.5063	0.3566	0.1877	0.10	1.9580
T_m	0.9898	0.9592	0.9083	0.8372	0.7461	0.6352	0.5048	0.3552	0.1868		
T_m/T	0.9999	0.9998	0.9994	0.9990	0.9984	0.9977	0.9969	0.9960	0.9949		
T	0.9887	0.9550	0.8993	0.8225	0.7257	0.6103	0.4780	0.3308	0.1707	1.0	1.7554
T_m	0.9888	0.9553	0.9000	0.8236	0.7271	0.6118	0.4794	0.3318	0.1712		
T_m/T	1.0001	1.0004	1.0008	1.0013	1.0019	1.0025	1.0029	1.0031	1.0028		
T	0.9851	0.9410	0.8698	0.7750	0.6610	0.5331	0.3971	0.2591	0.1249	10.0	1.1854
T_m	0.9860	0.9444	0.8766	0.7853	0.6736	0.5462	0.4082	0.2662	0.1273		
T_m/T	1.0009	1.0036	1.0078	1.0132	1.0191	1.0245	1.0280	1.0274	1.0196		
T	0.9838	0.9361	0.8597	0.7588	0.6393	0.5077	0.3711	0.2366	0.1110	101.0	1.0207
T_m	0.9852	0.9412	0.8699	0.7742	0.6582	0.5272	0.3876	0.2472	0.1146		
T_m/T	1.0014	1.0054	1.0119	1.0202	1.0296	1.0384	1.0447	1.0448	1.0332		

TABLE 3. Comparison of the Distributions of Dimensionless Temperatures according to [3] (T) and by the Method Proposed (T_m) in a Cylinder

Parameters	Values of parameters										Bi	A
ξ	0.9056	0.6792	0.4528	0.2264	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.10	1.9898
T	0.1801	0.5362	0.7930	0.9482	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
T_m	0.1792	0.5374	0.9741	0.9485	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
T_m/T	0.9946	1.0022	1.0015	1.0003	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		
ξ	0.9556	0.7963	0.6370	0.4778	0.3185	0.1593	0.0000	0.0000	0.0000	0.0000	1.0	1.7834
T	0.0792	0.3427	0.5697	0.7536	0.8891	0.9720	0.0000	0.0000	0.0000	0.0000		
T_m	0.0783	0.3408	0.5681	0.7526	0.8887	0.9720	0.0000	0.0000	0.0000	0.0000		
T_m/T	0.9883	0.9945	0.9971	0.9987	0.9995	1.0000	0.0000	0.0000	0.0000	0.0000		
ξ	0.9176	0.8259	0.7341	0.6423	0.5506	0.4588	0.3671	0.2753	0.1853	0.0918	10.0	1.3891
T	0.1166	0.2487	0.3801	0.5070	0.6256	0.7327	0.8250	0.8998	0.9549	0.9886		
T_m	0.1173	0.2517	0.3852	0.5134	0.6323	0.7387	0.8297	0.9028	0.9564	0.9890		
T_m/T	1.0064	1.0119	1.0134	1.0125	1.0107	1.0082	1.0056	1.0033	1.0015	1.0004		
ξ	0.8400	0.7560	0.6720	0.5880	0.5040	0.4200	0.3360	0.2520	0.1680	0.0840	100.0	1.2610
T	0.2141	0.3316	0.4485	0.5614	0.6669	0.7622	0.8444	0.9109	0.9599	0.9899		
T_m	0.2176	0.3380	0.4569	0.5707	0.6760	0.7699	0.8501	0.9145	0.9616	0.9904		
T_m/T	1.0166	1.0190	1.0187	1.0165	1.0135	1.0101	1.0068	1.0040	1.0018	1.0005		

$$\frac{T_1 - T_{1S}}{T_{10} - T_{1S}} = \frac{U(\mu_1 \cdot \bar{\xi}) - U(\mu_1 \cdot 1)}{U(\mu_1 \cdot 0) - U(\mu_1 \cdot 1)}, \tag{17}$$

$$A = \mathrm{Bi}\frac{U(\mu_1 \cdot 1)}{U(\mu_1 \cdot 0) - U(\mu_1 \cdot 1)}. \tag{18}$$

Consequently, out of the two parameters Fo and Bi, only the latter, whose real variation region according to the data of [3] is $\mathrm{Bi} = 10^{-2}-10^{2}$, is left in (17) and (18).

Comparison of the classical and proposed distributions of T_1 is presented in Tables 1-3 for five Biot numbers: 10^{-2}, 10^{-1}, 10^{0}, 10^{1}, and 10^{2} and for $\bar{\xi} = 0.1-0.9$ in 0.1. The use of the tabulated Bessel functions (Tables 2.11 and 2.13 in [3]) as basic values brought about the necessity to calculate ξ from them and to eliminate the case $\mathrm{Bi} = 10^{-2}$ in calculating the distributions of T_1 in a cylinder. The functions calculated by (14) and (17) are denoted by T, while the functions calculated by (8) and (12) are denoted by T_m. The ratios T_m/T are also presented.

The values of T, T_m, and T_m/T in Tables 1-3, in which data for $\mathrm{Fo} \geq 0.3$ are presented, need no discussion. Only a very narrow region of possible values of parameter A – from 1.021 to 1.996 – in classical solutions (Table 2) is noteworthy. In other words, classical solutions reflect a very "weak" external action on the object. An increase in parameter A is possible in classical solutions only when $\mathrm{Fo} < 0.3$ and hence makes us use formulas (14) and (15) when $n \gg 1$. In this case we should take into account that the condition $\mathrm{Fo} \to 0$ requires $n \to \infty$ be satisfied. In the method proposed, there are no similar difficulties.

4. In comparing the energy balance relations, we need data on the average temperatures of the solids that are calculated according to the formula

$$\hat{T}_1 = i \int_0^1 T_1 \eta^{i-1} d\eta \quad (i = 1, 2, 3). \tag{19}$$

Substitution of distributions (12) and (8) into (19) results in:

$$\text{when} \quad A > A_{\text{cr}} \quad \hat{T}_1 = \frac{AT_{10} + iT_{1S}}{A + i}, \tag{20}$$

$$\text{when} \quad A \le A_{\text{cr}} \quad \frac{\hat{T}_1 - T_{1S}}{T_{10} - T_{1S}} = b_0 + b_1 A, \tag{21}$$

where b_0 and b_1 are, respectively, 0.533 and 0.067 for $i = 1$, 0.333 and 0.083 for $i = 2$, and 0.229 and 0.086 for $i = 3$.

Problems when the external heat transfer agent is a mobile medium, whose average temperature differs little from T_{20}, are the most frequent. Therefore below we assume

$$\hat{T}_2 = T_{20}. \tag{22}$$

5. As a result of establishing relations (8), (9), (12), (16), and (20)-(22) the solution of the problem amounts to finding time and space variations in four functions T_{10}, $\hat{T}_1$, T_{1S}, and $\hat{T}_2$. Let us realize this part of the problem, using the following equations (in the form for the latter case).

a) *Equation of a variation in* T_{10} *that is determined by formula (3).* According to (12), when $A > A_{\text{cr}}$, we have ($A > 4$):

$$\left(\frac{\partial^2 T_1}{\partial \eta^2}\right)_0 = \lim_{\eta \to 0} \left[- (T_{10} - T_{1S}) A (A - 1) \eta^{A-2}\right] = 0. \tag{23}$$

Thus, when $A > A_{\text{cr}}$

$$\frac{dT_{10}}{dt} = 0, \tag{24}$$

which is quite explainable physically, since the temperature profile is very "filled" when $A > A_{\text{cr}}$ and this hinders diffusion of the heat from central regions of the solid. According to (3) and (8), when $A \le A_{\text{cr}}$, we have

$$\frac{dT_{10}}{dy} = - i \frac{2a_1}{\vec{v}_1 Z^2} [2 (T_{10} - T_{1S}) - 0.5 \,\text{Bi}\, (T_{1S} - \hat{T}_2)] \quad (i = 1, 2, 3). \tag{25}$$

If, as in classical problems, we set $\hat{T}_2 = \text{const}$ and assume a linear relationship between T_{1S} and T_{10} the solution of (25) will yield the known exponential law of a variation in T_{10}.

b) *Heat-balance equation*

$$\frac{d}{dy} (\vec{G}_1 c_{p1} \hat{T}_1 + \vec{G}_2 c_{p2} \hat{T}_2) = 0. \tag{26}$$

Here $\vec{G}_k$ ($k = 1, 2$) is interpreted as a vector whose positive direction coincides with the positive direction of y. Then (26) corresponds both to direct-flow and counterflow schemes of the interaction of phases. Equation (26) yields the relation

$$\hat{T}_2 = \frac{E(0) - \vec{G}_1 c_{p1} \hat{T}_1}{\vec{G}_2 c_{p2}}, \tag{27}$$

where

$$E(0) = \vec{G}_1 c_{p1}(0) \hat{T}_1(0) + \vec{G}_2 c_{p2}(0) \hat{T}_2(0). \tag{28}$$

c) *Interphase heat transfer equation*

$$\frac{d}{dy}(G_1 c_{p1} \hat{T}_1) = \alpha_2 (\hat{T}_2 - T_{1S}) b_{12}, \tag{29}$$

where b_{12} is the interface per unit length (height) of the apparatus:

$$b_{12} = \frac{S_{12}^{(1)} n_1 V_a}{H_a}. \tag{30}$$

For a weak dependence of c_{p1} on temperature, according to (29), we have

$$\frac{d\hat{T}_1}{dy} = \frac{\alpha_2 b_{12}}{G_1 c_{p1}} (\hat{T}_2 - T_{1S}). \tag{31}$$

d) *Relations (20) and (21) that need to be solved for* T_{1S}. When $A > A_{cr}$ it will be

$$T_{1S} = 0.5F_1 - \sqrt{0.25F_1^2 - F_2}, \tag{32}$$

where

$$F_1 = (T_{10} + \hat{T}_1) + \frac{\mathrm{Bi}}{i}(T_{10} - \hat{T}_1), \quad F_2 = T_{10}\hat{T}_1 + \frac{\mathrm{Bi}}{i}(T_{10} - \hat{T}_1)\hat{T}_2 \quad (i = 1, 2, 3). \tag{33}$$

When $A \le A_{cr}$ we obtain

$$T_{1S} = \frac{\hat{T}_1 - b_0 T_{10} + b_1 \mathrm{Bi}\, \hat{T}_2}{1 - b_0 + b_1 \mathrm{Bi}}. \tag{34}$$

The values of b_0 and b_1 correspond to the values adopted in formula (21).

The system of Eqs. (24) (or (25)), (27), (31), and (32) (or (34)) is closed, and the solution of the Cauchy problem enables us to establish variations in the direction of y (or with time) in the four sought functions T_{10}, $\hat{T}_1$, T_{1S}, and $\hat{T}_2$, and in parameter A. The Biot numbers are found in simultaneous solution of the hydrodynamical problem.

6. A change from thermal to diffusion problems of extraction of a substance from porous or capillary-porous solids is made by way of replacements:

$$T \to C, \quad a \to D, \quad \alpha_2 \to \beta_2, \quad \frac{\alpha_2 Z}{\lambda_1} \to \frac{\beta_2 Z}{D_1},$$
$$\vec{G}_k c_{pk} \to \vec{Q}_k \quad (k = 1, 2). \tag{35}$$

We should recognize two approaches in determining the concentration C_1 and the mass conductivity D_1. One correlates C_1 and D_1 with the entire volume of the solid or, alternatively, with only a porous portion of the solid filled with a substance carrier:

$$\hat{B}_1 = \frac{1}{V} \int_V B_1 dV. \tag{36}$$

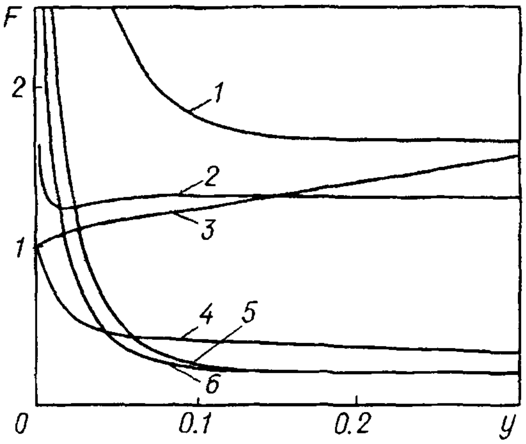

Fig. 1. Variation along the length of a heat exchanger in the functions: 1) $\mathrm{Bi}\cdot 10$, 2) A, 3) $p_2\cdot 10^{-5}$, 4) $u_{2f}\cdot 10^{-2}$, 5) $\hat{T}_1\cdot 10^{-2}$, 6) $\hat{T}_2\cdot 10^{-2}$.

In the majority of cases, the pore size is several decimal orders of magnitude smaller than the solid dimension. Then the porosity can be considered homogeneous and isotropic

$$B_1 \approx \hat{B}_1 = \text{const}. \tag{37}$$

According to (37), in the (26)-(31)-type relations, the replacement

$$\vec{G}c_{p1} \to \vec{Q}_1\hat{B}_1. \tag{38}$$

is required.

The conceptual advantage of the second approach is that D_1 turns out to be rigidly bound with the diffusion factor of the substance in a medium that fills the pores. Since this factor is known, as a rule, distorted interpretations of the values of D_1 drop out. A relationship between the quantities at the interface is somewhat more complex. This relationship can be expressed as a first approximation by the relation

$$C_{2S} = \psi_{21}B_{1S}C_{1S}. \tag{39}$$

The coefficient of the equilibrium distribution of the substance ψ_{21} is introduced for the cases when substance carriers in the solid and in the external medium are dissimilar and mutually insoluble. If the media are similar, $\psi_{21} = 1$. The coefficient B_{1S} reflects an inevitable concentration jump at the boundary of a porous solid with a continuous homogeneous phase. Thus, in the problems of diffusion extraction in accordance with the above the basic parameters can be represented as:

$$A \to A^* = \mathrm{Bi}^*\frac{C_{2S} - C_{20}}{C_{10} - C_{1S}}, \quad \mathrm{Bi} \to \mathrm{Bi}^* = \frac{\beta_2 Z}{B_{1S}D_1}. \tag{40}$$

Relation (39) will also have an effect on (33)- and (34)-type formulas that must have the form:

$$\begin{gathered} F_1 = (C_{10} + \hat{C}_1) + \frac{\mathrm{Bi}^*\psi_{21}B_{1S}}{i}(C_{10} - \hat{C}_1), \\ F_2 = C_{10}\hat{C}_1 + \frac{\mathrm{Bi}^*}{i}(C_{10} - \hat{C}_1)\hat{C}_2, \\ C_{1S} = \frac{\hat{C}_1 - b_0C_{10} + b_1\mathrm{Bi}^*\hat{C}_2}{1 - b_0 + b_1\mathrm{Bi}^*\psi_{21}B_{1S}}. \end{gathered} \tag{41}$$

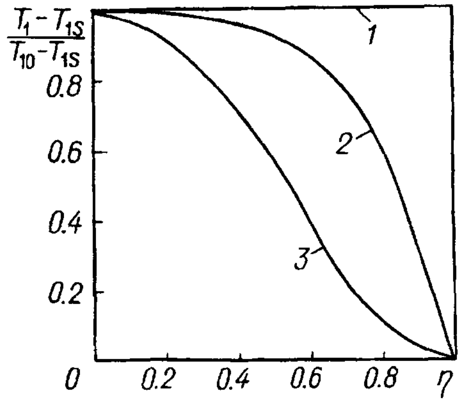

Fig. 2. Profiles of the excess temperatures: 1) $A \to \infty$, 2) 4, 3) 0.

Unlike heat problems, the solution of the problem of extraction of a substance amounts to finding five functions, i.e., C_{10}, $\hat{C}_1$, C_{1S}, C_{2S}, and $\hat{C}_2$ rather than four. The fifth, closing, relation is (39).

7. The example of solving the heat problem is presented by a calculation of continuous counterflow air cooling of a slowly moving packet of steel spheres (Fig. 1). The initial conditions for the packet are: $\rho_1 = 7.8 \cdot 10^3$ kg/m^3; $d_1 = 10^{-2}$ m; $\varphi_1 = 0.5$; $\hat{T}_1(0) = 400^\circ$C; $\vec{v}_1 = 10^{-2}$ m/sec; $S = 1$ m^2; $\vec{G}_1 = 39$ kg/sec; $b_{12} = 600$ m; $Z = 5 \cdot 10^{-3}$ m; for the air: $\vec{u}_{2f}(0) = -100$ m/sec; $\vec{G}_2 = -29.13$ kg/sec; $\hat{T}_2(0) = 325^\circ$C; $p_2(0) = 10^5$ Pa.

Thermophysical properties of the steel and the air were approximated by linear temperature dependences; the air density was calculated by the Clapeyron equation. Aerodynamic parameters and the heat transfer coefficients were determined by the procedure presented in [4]. Although the cooling was in essence completed at a length of 0.1 m, the temperatures in the body of the spheres were totally equalized (accurate up to 10^{-2} $^\circ$C) in the cross section offset by 0.32 m from the initial cross section. The aerodynamic characteristics (pressure, velocity in narrow cross sections) varied substantially along the heat-exchanger length, which resulted in adequate change in the Biot number. A sharp decrease in the values at the beginning of the heat exchanger with a tendency toward an abrupt increase in the end is characteristic of parameter A. The steepness of temperature curves made it impossible to draw the temperatute lines T_{10} and T_{1S} on the plot. The region of variation in the ratios $\hat{T}_1/T_{10}$ was bounded below by a value of 0.926 and by a value of 0.889 for the ratios T_{1S}/T_{10}. Figure 2 gives an idea of a variation in the excess temperatures.

The example of solving the problem of diffusion extraction is presented by a calculation of the aqueous-solution extraction of saccharose from beet chips in a KPA-30 counterflow column diffusion apparatus with a capacity of up to 3,000 tons of chips a day [5]. Saccharose in beets is initially present as an aqueous solution ($\hat{C}_1(0) = C_{10} = C_{1S} = 220$ kg/m^3) that fills up to 90% of the beets volume ($\hat{B}_1 = 0.9$, $\psi_{21} = 1$). The beets are cut into long diamond-shaped chips with an equivalent diameter $d_1 = 3 \cdot 10^{-3}$ m ($Z = 1.5 \cdot 10^{-3}$ m, $B_{1S} = 0.9$). In the column apparatus, the volumetric rates of the beets and the aqueous solution are magnitudes of the order $\vec{Q}_1 = 35 \cdot 10^{-3}$ m^3/sec and $\vec{Q}_2 = -45 \cdot 10^{-3}$ m^3/sec. For the active height of the apparatus $H_a = 13$ m and the time of stay of the beets in it $\tau = 80$ min; the velocity will be $\vec{v}_1 = 2.71 \cdot 10^{-3}$ m/sec. The specific load of the working volume of the apparatus with the chips is 700 kg/m^3. Then the velocity of extragent motion is:

$$\vec{v}_2 = \frac{\vec{Q}_2 u_1 \cdot 0.7}{\vec{Q}_1 \cdot 0.3} = -8.13 \cdot 10^{-3} \text{ m/sec},$$

and the relative phase velocity is $u_{21} = 10.84 \cdot 10^{-3}$ m/sec. The counting bulk density of cylinders more than 1 m long is $n_1 = 1.11 \cdot 10^5$ m^{-3}. According to (30), for a useful volume of the apparatus of 208 m^3 we obtain $b_{12} = 1.67 \cdot 10^4$ m. The mass conductivity in the solution that fills the chips is taken to be equal to the factor of saccharose diffusion in water $D_1 \approx 0.5 \cdot 10^{-9}$ m^2/sec. The kinematic viscosity factor of the extragent is $\nu_2 \approx 0.7 \cdot 10^{-6}$ m^2/sec. We calculate the mass transfer coefficient by the formula

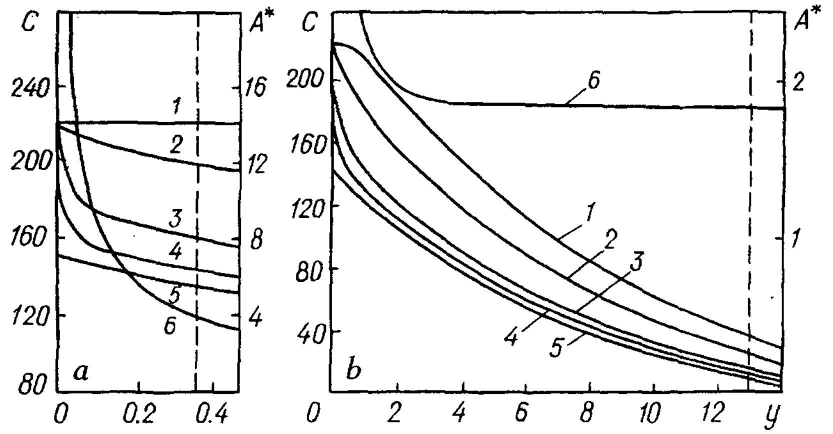

Fig. 3. Variation in the concentrations and in the parameter A^* on the initial segment (a) and along the column height (b): 1) C_{10}, 2) $\hat{C}_1$, 3) C_{1S}, 4) C_{2S}, 5) C_{20}, 6) A^*.

TABLE 4. Comparison of the Thermal Parameters according to [6] and by the Method Proposed

Bi	θ	Source	Fo							
			1	2	3	4	5	6	10	12
0.1	θ_w	[6]	0.076	0.119	0.160	0.198	0.232	0.268	0.380	0.425
		Our data	0.076	0.119	0.160	0.197	0.233	0.266	0.380	0.428
	θ_c	[6]	0.033	0.085	0.119	0.160	0.198	0.232	0.350	0.402
		Our data	0.033	0.077	0.119	0.158	0.195	0.230	0.349	0.399
0.4	θ_w	[6]	0.257	0.370	0.450	0.525	0.586	0.648	0.750	0.800
		Our data	0.253	0.367	0.456	0.528	0.587	0.634	0.758	0.795
	θ_c	[6]	0.112	0.242	0.348	0.440	0.503	0.560	0.702	0.750
		Our data	0.115	0.244	0.349	0.434	0.504	0.561	0.709	0.754

$$\beta_2 = 0.8\,\mathrm{Pr}_2^{0.33}\,\mathrm{Re}_2^{0.5}\,\frac{D_1}{d_1},$$

where $\mathrm{Pr}_2 = \nu_2/D_1$; $\mathrm{Re}_2 = u_{21}d_1/\nu_2$. For $\hat{C}_1(0) = 220$ kg/m^3, $\hat{C}_2(0) = 150$ kg/m^3. The process of extraction in the KDA-30 apparatus is presented in Fig. 3. The region of $A < A_{\mathrm{cr}}$ is small and is bounded by $y_{\mathrm{cr}} = 0.34$ m (Fig. 3a), and the initially rectangular profile of the excess concentrations on this segment becomes a profile with $A^* = A^*_{\mathrm{cr}}$ (see Fig. 2 as applied to diffusion extraction). When $y > y_{\mathrm{cr}}$ there occurs a monotonic decrease in five basic functions: C_{10}, $\hat{C}_1$, C_{1S}, C_{2S}, and $\hat{C}_2$, whose values at the outlet from the apparatus correspond to practical data. Parameter A^* is characterized by an asymptotic approximation to a value of 1.839 for $y = 4.74$ m. (The vertical dashed line in Fig. 3a corresponds to A^*_{cr}.)

As follows from the above examples the method proposed is free of any restrictions on the character of space-time variation in the entire set of external and boundary conditions. Known approximate solutions are obtained for the cases when only one parameter, for example, the temperature of the external medium, varies according to one of the three laws: linear, harmonic, and exponential. Table 4 compares the results of calculations by this procedure and on the basis of integral Laplace transforms with the use of variational methods [6] for a plate with linear variation in the temperature of the external medium. The parameters in Table 4 correspond to identically marked parameters of Tables 1-2 of [6] and reflect the variations in the dimensionless temperatures θ_w and θ_c, respectively, on the wall and at the center of the plate as functions of the Fourier and Biot numbers.

Thus, we can infer that the method proposed for solving unsteady problems of heat and mass conduction with varying boundary conditions makes it possible to obtain mathematical relations with a "transparent" physical interpretation and simple computational algorithms.

NOTATION

a_k, polynomial coefficients; a, thermal diffusivity, m^2/sec; A and A^*, parameters of steepness of the temperature and substance distributions in the solid; b_{12}, interface per unit height of the apparatus, m; c_p, specific heat, J/(kg·K); C, concentration of the substance, kg/m^3; d, diameter of the packet element, m; D, mass conductivity, m^2/sec; G, mass flow rate, kg/sec; H, height of the apparatus, m; Q, volumetric rate, m^3/sec; i, natural number ($i = 1$ is the plate, $i = 2$ is the cylinder, and $i = 3$ is the sphere); n, counting bulk density of the packet elements, m^{-3}; p, pressure, Pa; S, surface area of the solid, cross-sectional area, m^2; t, time, sec; T, temperature, °C; u_{2f}, velocity in a narrow cross section of the packet, m/sec; v, velocity, m/sec; V, volume, m^3; y, longitudinal coordinate, m; z, coordinate counted from the surface, the line, and the point of symmetry, m; Z, characteristic half-dimension of the solid, m; α, heat-transfer coefficient, W/(m^2·K); β, mass-transfer coefficient, m/sec; B, indicator function of the porosity of particles (the analog of the bulk concentration of pores); φ, porosity of the packet; λ, thermal conductivity, W/(m·K); ψ, equilibrium coefficient of the substance distribution between carriers in phases; ρ, density, kg/m^3; τ, time interval, sec. Subscripts and superscripts: 0, value at $z = 0$ or in the volume of phase 2; 1, 2, phases 1 and 2; 1 2, between phases 1 and 2; (1) unit element; S, at the interface; a, apparatus; f, narrow cross section of the packet; $\hat{}$, average value; cr, critical; m, model.

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